ON SOME HADAMARD-TYPE INEQUALITIES FOR CO-ORDINATED CONVEX FUNCTIONS

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Abstract. In this paper, we prove some new inequalities of Hadamard-type for convex functions on the co-ordinates.

1. INTRODUCTION

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a < b$. The following double inequality;

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}$$

is well known in the literature as Hadamard's inequality. Both inequalities hold in the reversed direction if f is concave.

In [1], Dragomir defined convex functions on the co-ordinates as following:

Definition 1. *Let us consider the bidimensional interval $\Delta = [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$, $c < d$. A function $f : \Delta \to \mathbb{R}$ will be called convex on the co-ordinates if the partial mappings $f_y : [a,b] \to \mathbb{R}$, $f_y(u) = f(u,y)$ and $f_x : [c,d] \to \mathbb{R}$, $f_x(v) = f(x,v)$ are convex where defined for all $y \in [c,d]$ and $x \in [a,b]$. Recall that the mapping $f : \Delta \to \mathbb{R}$ is convex on Δ if the following inequality holds,*

$$f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \leq \lambda f(x,y) + (1-\lambda) f(z,w)$$

for all $(x,y),(z,w) \in \Delta$ and $\lambda \in [0,1]$.

In [1], Dragomir established the following inequalities of Hadamard's type for co-ordinated convex functions on a rectangle from the plane $\mathbb{R}^2$.

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Theorem 1. *Suppose that $f:\Delta=[a,b]\times[c,d]\to\mathbb{R}$ is convex on the co-ordinates on Δ. Then one has the inequalities;*

$$
\begin{aligned}
&f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \\
\leq\ & \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right] \\
\leq\ & \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy \\
\leq\ & \frac{1}{4}\left[\frac{1}{(b-a)}\int_a^b f(x,c)dx+\frac{1}{(b-a)}\int_a^b f(x,d)dx\right. \\
& \left.+\frac{1}{(d-c)}\int_c^d f(a,y)dy+\frac{1}{(d-c)}\int_c^d f(b,y)dy\right] \\
\leq\ & \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.
\end{aligned}
\tag{1.1}
$$

The above inequalities are sharp.

Similar results can be found in [1]-[7].

The main purpose of this paper is to prove some new inequalities of Hadamard-type for convex functions on the co-ordinates.

2. MAIN RESULTS

To prove our main result, we need the following Lemma.

Lemma 1. *Let $f:\Delta=[a,b]\times[c,d]\to\mathbb{R}$ be a twice partial differentiable mapping on $\Delta=[a,b]\times[c,d]$. If $\frac{\partial^2 f}{\partial t\partial s}\in L(\Delta)$, then the following equality holds:*

$$
\begin{aligned}
& A+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(u,v)\,dudv \\
=\ & \frac{(x-a)^2(y-c)^2}{(b-a)(d-c)}\int_0^1\int_0^1 (t-1)(s-1)\frac{\partial^2 f}{\partial t\partial s}(tx+(1-t)a,sy+(1-s)c)\,dsdt \\
& +\frac{(x-a)^2(d-y)^2}{(b-a)(d-c)}\int_0^1\int_0^1 (t-1)(1-s)\frac{\partial^2 f}{\partial t\partial s}(tx+(1-t)a,sy+(1-s)d)\,dsdt \\
& +\frac{(b-x)^2(y-c)^2}{(b-a)(d-c)}\int_0^1\int_0^1 (1-t)(s-1)\frac{\partial^2 f}{\partial t\partial s}(tx+(1-t)b,sy+(1-s)c)\,dsdt \\
& +\frac{(b-x)^2(d-y)^2}{(b-a)(d-c)}\int_0^1\int_0^1 (1-t)(1-s)\frac{\partial^2 f}{\partial t\partial s}(tx+(1-t)b,sy+(1-s)d)\,dsdt
\end{aligned}
$$

where

$$\begin{aligned} A = & \frac{(x-a)(y-c) f(a,c) + (x-a)(d-y) f(a,d)}{(b-a)(d-c)} \\ & + \frac{(b-x)(y-c) f(b,c) + (b-x)(d-y) f(b,d)}{(b-a)(d-c)} \\ & - \frac{(x-a)}{(b-a)(d-c)} \int_c^d f(a,v)\,dv - \frac{(b-x)}{(b-a)(d-c)} \int_c^d f(b,v)\,dv \\ & - \frac{(d-y)}{(b-a)(d-c)} \int_a^b f(u,d)\,du - \frac{(y-c)}{(b-a)(d-c)} \int_a^b f(u,c)\,du \end{aligned}$$

Proof. It suffices to note that

$$\begin{aligned} I = & \underbrace{\frac{(x-a)^2 (y-c)^2}{(b-a)(d-c)} \int_0^1 \int_0^1 (t-1)(s-1) \frac{\partial^2 f}{\partial t \partial s} (tx + (1-t) a, sy + (1-s) c)\, dsdt}_{I_1} \\ & \underbrace{+ \frac{(x-a)^2 (d-y)^2}{(b-a)(d-c)} \int_0^1 \int_0^1 (t-1)(1-s) \frac{\partial^2 f}{\partial t \partial s} (tx + (1-t) a, sy + (1-s) d)\, dsdt}_{I_2} \\ & \underbrace{+ \frac{(b-x)(y-c)}{(b-a)(d-c)} \int_0^1 \int_0^1 (1-t)(s-1) \frac{\partial^2 f}{\partial t \partial s} (tx + (1-t) b, sy + (1-s) c)\, dsdt}_{I_3} \\ & \underbrace{+ \frac{(b-x)(d-y)}{(b-a)(d-c)} \int_0^1 \int_0^1 (1-t)(1-s) \frac{\partial^2 f}{\partial t \partial s} (tx + (1-t) b, sy + (1-s) d)\, dsdt.}_{I_4} \end{aligned}$$

Integrating by parts, we get

$$\begin{aligned} I_1 = & \frac{(x-a)^2 (y-c)^2}{(b-a)(d-c)} \int_0^1 (s-1) \left[\frac{t-1}{x-a} \frac{\partial f}{\partial s} (tx + (1-t) a, sy + (1-s) c) \Big|_0^1 \right. \\ & \left. - \frac{1}{x-a} \int_0^1 \frac{\partial f}{\partial s} (tx + (1-t) a, sy + (1-s) c)\, dt \right] ds \\ = & \frac{(x-a)^2 (y-c)^2}{(b-a)(d-c)} \int_0^1 (s-1) \left[\frac{1}{x-a} \frac{\partial f}{\partial s} (a, sy + (1-s) c) \right. \\ & \left. - \frac{1}{x-a} \int_0^1 \frac{\partial f}{\partial s} (tx + (1-t) a, sy + (1-s) c)\, dt \right] ds \end{aligned}$$

By integrating again and by changing of the variables $u = tx + (1-t)a$, $v = sy + (1-s)c$, we obtain

$$\begin{aligned} I_1 &= \frac{(x-a)(y-c)}{(b-a)(d-c)} \int_0^1 (s-1) \left[\frac{t-1}{x-a} \frac{\partial f}{\partial s} (tx + (1-t)a, sy + (1-s)c) \Big|_0^1 \right. \\ & \left. - \frac{1}{x-a} \int_0^1 \frac{\partial f}{\partial s} (tx + (1-t)a, sy + (1-s)c) \, dt \right] ds \\ &= \frac{1}{(x-a)(y-c)} f(a,c) - \frac{1}{(x-a)(y-c)^2} \int_c^y f(a,v) \, dv \\ & - \frac{1}{(x-a)^2 (y-c)} \int_a^x f(u,c) \, du + \frac{1}{(x-a)^2 (y-c)^2} \int_a^x \int_c^y f(u,v) \, du dv. \end{aligned}$$

By a similar argument, we have

$$\begin{aligned} I_2 &= \frac{1}{(x-a)(d-y)} f(a,d) - \frac{1}{(x-a)(d-y)^2} \int_y^d f(a,v) \, dv \\ & - \frac{1}{(x-a)^2 (d-y)} \int_a^x f(u,d) \, du \\ & + \frac{1}{(x-a)^2 (d-y)^2} \int_a^x \int_y^d f(u,v) \, du dv, \end{aligned}$$

$$\begin{aligned} I_3 &= \frac{1}{(b-x)(y-c)} f(b,c) - \frac{1}{(b-x)(y-c)^2} \int_c^y f(b,v) \, dv \\ & - \frac{1}{(b-x)^2 (y-c)} \int_x^b f(u,c) \, du \\ & + \frac{1}{(b-x)^2 (y-c)^2} \int_x^b \int_c^y f(u,v) \, du dv \end{aligned}$$

and

$$
\begin{aligned}
I_4 &= \frac{1}{(b-x)(d-y)} f(b,d) - \frac{1}{(b-x)(d-y)^2} \int_y^d f(b,v)\,dv \\
&\quad - \frac{1}{(b-x)^2(d-y)} \int_x^b f(u,d)\,du \\
&\quad + \frac{1}{(b-x)^2(d-y)^2} \int_x^b \int_y^d f(u,v)\,du dv.
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
& I_1 + I_2 + I_3 + I_4 \\
= \; & \frac{1}{(b-a)(d-c)} \\
& \times \left[A - (x-a) \left[\int_y^d f(a,v)\,dv + \int_c^y f(a,v)\,dv \right] - (b-x) \left[\int_c^y f(b,v)\,dv + \int_y^d f(b,v)\,dv \right] \right. \\
& - (d-y) \left[\int_a^x f(u,d)\,du + \int_x^b f(u,d)\,du \right] - (y-c) \left[\int_a^x f(u,c)\,du + \int_x^b f(u,c)\,du \right] \\
& \left. + \int_x^b \int_c^y f(u,v)\,du dv + \int_x^b \int_y^d f(u,v)\,du dv + \int_a^x \int_c^y f(u,v)\,du dv + \int_a^x \int_y^d f(u,v)\,du dv \right. \\
= \; & \frac{1}{(b-a)(d-c)} \left[A - (x-a) \int_c^d f(a,v)\,dv - (b-x) \int_c^d f(b,v)\,dv \right. \\
& \left. - (d-y) \int_a^b f(u,d)\,du - (y-c) \int_a^b f(u,c)\,du + \int_a^b \int_c^d f(u,v)\,du dv. \right.
\end{aligned}
$$

Which completes the proof. $\square$

Theorem 2. *Let $f : \Delta = [a,b] \times [c,d] \to \mathbb{R}$ be a partial differentiable mapping on $\Delta = [a,b] \times [c,d]$ and $\frac{\partial^2 f}{\partial t \partial s} \in L(\Delta)$. If $\left| \frac{\partial^2 f}{\partial t \partial s} \right|$ is a convex function on the co-ordinates*

on Δ, *then the following inequality holds;*

$$
\begin{aligned}
&\left|A+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(u,v)\,dudv\right| \\
\leq\ & \frac{1}{9(b-a)(d-c)}\left[\left(\frac{\left((x-a)^2+(b-x)^2\right)\left((y-c)^2+(d-y)^2\right)}{4}\right)\left|\frac{\partial^2 f}{\partial t\partial s}(x,y)\right|\right. \\
&+\left(\frac{(x-a)^2\left((y-c)^2+(d-y)^2\right)}{2}\right)\left|\frac{\partial^2 f}{\partial t\partial s}(a,y)\right| \\
&+\left(\frac{(b-x)^2\left((y-c)^2+(d-y)^2\right)}{2}\right)\left|\frac{\partial^2 f}{\partial t\partial s}(b,y)\right| \\
&+\left(\frac{(y-c)^2\left((x-a)^2+(b-x)^2\right)}{2}\right)\left|\frac{\partial^2 f}{\partial t\partial s}(x,c)\right| \\
&+\left(\frac{(d-y)^2\left((x-a)^2+(b-x)^2\right)}{2}\right)\left|\frac{\partial^2 f}{\partial t\partial s}(x,d)\right| \\
&+(x-a)^2(y-c)^2\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|+(x-a)^2(d-y)^2\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right| \\
&\left.+(b-x)^2(y-c)^2\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|+(b-x)^2(d-y)^2\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|\right].
\end{aligned}
$$

Proof. From Lemma 1 and using the property of modulus, we have

$$
\begin{aligned}
&\left|A+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(u,v)\,dudv\right| \\
\leq\ & \frac{(x-a)^2(y-c)^2}{(b-a)(d-c)}\int_0^1\int_0^1 |(t-1)(s-1)|\left|\frac{\partial^2 f}{\partial t\partial s}(tx+(1-t)a,sy+(1-s)c)\right|dsdt \\
&+\frac{(x-a)^2(d-y)^2}{(b-a)(d-c)}\int_0^1\int_0^1 |(t-1)(1-s)|\left|\frac{\partial^2 f}{\partial t\partial s}(tx+(1-t)a,sy+(1-s)d)\right|dsdt \\
&+\frac{(b-x)^2(y-c)^2}{(b-a)(d-c)}\int_0^1\int_0^1 |(1-t)(s-1)|\left|\frac{\partial^2 f}{\partial t\partial s}(tx+(1-t)b,sy+(1-s)c)\right|dsdt \\
&+\frac{(b-x)^2(d-y)^2}{(b-a)(d-c)}\int_0^1\int_0^1 |(1-t)(1-s)|\left|\frac{\partial^2 f}{\partial t\partial s}(tx+(1-t)b,sy+(1-s)d)\right|dsdt.
\end{aligned}
$$

Since $\left|\frac{\partial^2 f}{\partial t \partial s}\right|$ is co-ordinated convex, we can write

$$
\begin{aligned}
&\left|A+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(u,v)\,dudv\right| \\
\leq\ & \frac{(x-a)^2(y-c)^2}{(b-a)(d-c)}\int_0^1 |(s-1)| \\
&\times\left[\int_0^1 (t-1)\,t\left|\frac{\partial^2 f}{\partial t\partial s}(x,sy+(1-s)c)\right|dt\right. \\
&\left.+\int_0^1 (t-1)(1-t)\left|\frac{\partial^2 f}{\partial t\partial s}(a,sy+(1-s)c)\right|dt\right]ds \\
&+\frac{(x-a)^2(d-y)^2}{(b-a)(d-c)}\int_0^1 |(1-s)| \\
&\times\left[\int_0^1 (t-1)\,t\left|\frac{\partial^2 f}{\partial t\partial s}(x,sy+(1-s)d)\right|dt\right. \\
&\left.+\int_0^1 (t-1)(1-t)\left|\frac{\partial^2 f}{\partial t\partial s}(a,sy+(1-s)d)\right|dt\right]ds \\
&+\frac{(b-x)^2(y-c)^2}{(b-a)(d-c)}\int_0^1 |(s-1)| \\
&\times\left[\int_0^1 (t-1)\,t\left|\frac{\partial^2 f}{\partial t\partial s}(x,sy+(1-s)c)\right|dt\right. \\
&\left.+\int_0^1 (t-1)(1-t)\left|\frac{\partial^2 f}{\partial t\partial s}(b,sy+(1-s)c)\right|dt\right]ds \\
&+\frac{(b-x)^2(d-y)^2}{(b-a)(d-c)}\int_0^1 |(1-s)| \\
&\times\left[\int_0^1 (1-t)\,t\left|\frac{\partial^2 f}{\partial t\partial s}(x,sy+(1-s)d)\right|dt\right. \\
&\left.+\int_0^1 (1-t)(1-t)\left|\frac{\partial^2 f}{\partial t\partial s}(b,sy+(1-s)d)\right|dt\right]ds.
\end{aligned}
$$

By computing these integrals, we obtain

$$\begin{aligned}
&\left|A+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(u,v)\,dudv\right| \\
\leq\ & \frac{(x-a)^2(y-c)^2}{(b-a)(d-c)}\int_0^1 |s-1|\left[-\frac{1}{6}\left|\frac{\partial^2 f}{\partial t\partial s}(x,sy+(1-s)c)\right|-\frac{1}{3}\left|\frac{\partial^2 f}{\partial t\partial s}(a,sy+(1-s)c)\right|\right]ds \\
&+\frac{(x-a)^2(d-y)^2}{(b-a)(d-c)}\int_0^1 |1-s|\left[-\frac{1}{6}\left|\frac{\partial^2 f}{\partial t\partial s}(x,sy+(1-s)d)\right|-\frac{1}{3}\left|\frac{\partial^2 f}{\partial t\partial s}(a,sy+(1-s)d)\right|\right]ds \\
&+\frac{(b-x)^2(y-c)^2}{(b-a)(d-c)}\int_0^1 |s-1|\left[-\frac{1}{6}\left|\frac{\partial^2 f}{\partial t\partial s}(x,sy+(1-s)c)\right|-\frac{1}{3}\left|\frac{\partial^2 f}{\partial t\partial s}(b,sy+(1-s)c)\right|\right]ds \\
&+\frac{(b-x)^2(d-y)^2}{(b-a)(d-c)}\int_0^1 |1-s|\left[-\frac{1}{6}\left|\frac{\partial^2 f}{\partial t\partial s}(x,sy+(1-s)d)\right|-\frac{1}{3}\left|\frac{\partial^2 f}{\partial t\partial s}(b,sy+(1-s)d)\right|\right]ds.
\end{aligned}$$

Using co-ordinated convexity of $\left|\frac{\partial^2 f}{\partial t\partial s}\right|$ again and computing all integrals, we obtain

$$\begin{aligned}
&\left|A+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(u,v)\,dudv\right| \\
\leq\ & \frac{1}{9(b-a)(d-c)}\left[\left(\frac{\left((x-a)^2+(b-x)^2\right)\left((y-c)^2+(d-y)^2\right)}{4}\right)\left|\frac{\partial^2 f}{\partial t\partial s}(x,y)\right|\right. \\
&+\left(\frac{(x-a)^2\left((y-c)^2+(d-y)^2\right)}{2}\right)\left|\frac{\partial^2 f}{\partial t\partial s}(a,y)\right| \\
&+\left(\frac{(b-x)^2\left((y-c)^2+(d-y)^2\right)}{2}\right)\left|\frac{\partial^2 f}{\partial t\partial s}(b,y)\right| \\
&+\left(\frac{(y-c)^2\left((x-a)^2+(b-x)^2\right)}{2}\right)\left|\frac{\partial^2 f}{\partial t\partial s}(x,c)\right| \\
&+\left(\frac{(d-y)^2\left((x-a)^2+(b-x)^2\right)}{2}\right)\left|\frac{\partial^2 f}{\partial t\partial s}(x,d)\right| \\
&+(x-a)^2(y-c)^2\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|+(x-a)^2(d-y)^2\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right| \\
&\left.+(b-x)^2(y-c)^2\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|+(b-x)^2(d-y)^2\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|\right].
\end{aligned}$$

Which completes the proof. □

Corollary 1. *(1) Under the assumptions of Theorem 2, if we choose $x=a$, $y=c$, we obtain the following inequality;*

$$\begin{aligned}
&\left|\frac{f(b,d)}{(b-a)(d-c)}-\frac{1}{d-c}\int_c^d f(b,v)\,dv\right. \\
&\left.-\frac{1}{b-a}f(u,d)\,du+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(u,v)\,dudv\right| \\
\leq\ & \frac{1}{36(b-a)(d-c)}\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|+\frac{1}{9(b-a)(d-c)}\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right| \\
&+\frac{1}{18(b-a)(d-c)}\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|+\frac{1}{18(b-a)(d-c)}\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|.
\end{aligned}$$

(2) Under the assumptions of Theorem 2, if we choose $x=b$, $y=d$, we obtain the following inequality;

$$\begin{aligned}
&\left|\frac{f(a,c)}{(b-a)(d-c)}-\frac{1}{d-c}\int_c^d f(a,v)\,dv\right. \\
&\left.-\frac{1}{b-a}\int_a^b f(u,c)\,du+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(u,v)\,dudv\right| \\
\leq\ & \frac{(b-a)(d-c)}{36}\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|+\frac{1}{9(b-a)(d-c)}\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right| \\
&+\frac{(b-a)(d-c)}{18}\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|+\frac{(b-a)(d-c)}{18}\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|.
\end{aligned}$$

(3) Under the assumptions of Theorem 2, if we choose $x=\frac{a+b}{2}$, $y=\frac{c+d}{2}$, we obtain the following inequality;

$$\begin{aligned}
&\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4(b-a)(d-c)}-\frac{1}{2(d-c)}\int_c^d f(a,v)\,dv-\frac{1}{2(d-c)}\int_c^d f(b,v)\,dv\right. \\
&\left.-\frac{1}{2(b-a)}\int_a^b f(u,d)\,du-\frac{1}{2(b-a)}\int_a^b f(u,c)\,du+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(u,v)\,dudv\right| \\
\leq\ & \frac{1}{144(b-a)(d-c)}\left[\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}\left(a,\frac{c+d}{2}\right)\right|\right. \\
&+\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,\frac{c+d}{2}\right)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},c\right)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},d\right)\right| \\
&\left.+\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|\right].
\end{aligned}$$

Theorem 3. *Let $f:\Delta=[a,b]\times[c,d]\to\mathbb{R}$ be a partial differentiable mapping on $\Delta=[a,b]\times[c,d]$ and $\frac{\partial^2 f}{\partial t\partial s}\in L(\Delta)$. If $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q, q>1,$ is a convex function on the*

co-ordinates on Δ, *then the following inequality holds;*

$$\begin{aligned}
&\left|A+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(u,v)\,dudv\right| \\
\leq\ & \frac{1}{2^{\frac{2}{q}}(p+1)^{\frac{2}{p}}}\times \\
&\left\{\frac{(x-a)^2(y-c)^2}{(b-a)(d-c)}\left(\left|\frac{\partial^2 f}{\partial t\partial s}(x,y)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(x,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(a,y)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q\right)^{\frac{1}{q}}\right. \\
&+\frac{(x-a)^2(d-y)^2}{(b-a)(d-c)}\left(\left|\frac{\partial^2 f}{\partial t\partial s}(x,y)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(x,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(a,y)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q\right)^{\frac{1}{q}} \\
&+\frac{(b-x)^2(y-c)^2}{(b-a)(d-c)}\left(\left|\frac{\partial^2 f}{\partial t\partial s}(x,y)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(x,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,y)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q\right)^{\frac{1}{q}} \\
&\left.+\frac{(b-x)^2(d-y)^2}{(b-a)(d-c)}\left(\left|\frac{\partial^2 f}{\partial t\partial s}(x,y)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(x,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,y)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q\right)^{\frac{1}{q}}\right\}
\end{aligned}$$

where $p^{-1}+q^{-1}=1$.

Proof. From Lemma 1, we have

$$\begin{aligned}
&\left|A+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(u,v)\,dudv\right| \\
\leq\ & \frac{(x-a)^2(y-c)^2}{(b-a)(d-c)}\int_0^1\int_0^1 |(t-1)(s-1)|\left|\frac{\partial^2 f}{\partial t\partial s}(tx+(1-t)a,sy+(1-s)c)\right|dsdt \\
&+\frac{(x-a)^2(d-y)^2}{(b-a)(d-c)}\int_0^1\int_0^1 |(t-1)(1-s)|\left|\frac{\partial^2 f}{\partial t\partial s}(tx+(1-t)a,sy+(1-s)d)\right|dsdt \\
&+\frac{(b-x)^2(y-c)^2}{(b-a)(d-c)}\int_0^1\int_0^1 |(1-t)(s-1)|\left|\frac{\partial^2 f}{\partial t\partial s}(tx+(1-t)b,sy+(1-s)c)\right|dsdt \\
&+\frac{(b-x)^2(d-y)^2}{(b-a)(d-c)}\int_0^1\int_0^1 |(1-t)(1-s)|\left|\frac{\partial^2 f}{\partial t\partial s}(tx+(1-t)b,sy+(1-s)d)\right|dsdt.
\end{aligned}$$

By using the well known Hölder inequality for double integrals, then one has:

$$\begin{aligned}
&\left|A+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(u,v)\,dudv\right| \\
\leq\ & \frac{(x-a)^2(y-c)^2}{(b-a)(d-c)}\left(\int_0^1\int_0^1 |(t-1)(s-1)|^p\,dsdt\right)^{\frac{1}{p}} \\
&\times\left(\int_0^1\int_0^1\left|\frac{\partial^2 f}{\partial t\partial s}(tx+(1-t)a,sy+(1-s)c)\right|^q dsdt\right)^{\frac{1}{q}} \\
&+\frac{(x-a)^2(d-y)^2}{(b-a)(d-c)}\left(\int_0^1\int_0^1 |(t-1)(1-s)|^p\,dsdt\right)^{\frac{1}{p}} \\
&\times\left(\int_0^1\int_0^1\left|\frac{\partial^2 f}{\partial t\partial s}(tx+(1-t)a,sy+(1-s)d)\right|^q dsdt\right)^{\frac{1}{q}} \\
&+\frac{(b-x)^2(y-c)^2}{(b-a)(d-c)}\left(\int_0^1\int_0^1 |(1-t)(s-1)|^p\,dsdt\right)^{\frac{1}{p}} \\
&\times\left(\int_0^1\int_0^1\left|\frac{\partial^2 f}{\partial t\partial s}(tx+(1-t)b,sy+(1-s)c)\right|^q dsdt\right)^{\frac{1}{q}} \\
&+\frac{(b-x)^2(d-y)^2}{(b-a)(d-c)}\left(\int_0^1\int_0^1 |(1-t)(1-s)|^p\,dsdt\right)^{\frac{1}{p}} \\
&\times\left(\int_0^1\int_0^1\left|\frac{\partial^2 f}{\partial t\partial s}(tx+(1-t)b,sy+(1-s)d)\right|^q dsdt.\right)^{\frac{1}{q}}
\end{aligned}$$

Since $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$ is convex function on the co-ordinates on Δ, we know that for $t\in[0,1]$

$$\begin{aligned}
&\left|\frac{\partial^2 f}{\partial t\partial s}(tx+(1-t)a,sy+(1-s)c)\right|^q \\
\leq\ & t\left|\frac{\partial^2 f}{\partial t\partial s}(x,sy+(1-s)c)\right|^q+(1-t)\left|\frac{\partial^2 f}{\partial t\partial s}(a,sy+(1-s)c)\right|^q \\
\leq\ & t\left(s\left|\frac{\partial^2 f}{\partial t\partial s}(x,y)\right|^q+(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(x,c)\right|^q\right)+(1-t)\left(s\left|\frac{\partial^2 f}{\partial t\partial s}(a,y)\right|^q+(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q\right)
\end{aligned}$$

and by using the fact that

$$\left(\int_0^1\int_0^1 |(t-1)(s-1)|^p\,dsdt\right)^{\frac{1}{p}} = \frac{1}{(p+1)^{\frac{2}{p}}}$$

we get

$$\begin{aligned}
&\left(\int_0^1\int_0^1 \left|\frac{\partial^2 f}{\partial t\partial s}(tx+(1-t)a, sy+(1-s)c)\right|^q dsdt\right)^{\frac{1}{q}} \\
\leq\; &\left(\int_0^1\int_0^1 \left\{ts\left|\frac{\partial^2 f}{\partial t\partial s}(x,y)\right|^q + t(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(x,c)\right|^q\right.\right. \\
&\left.\left. + (1-t)s\left|\frac{\partial^2 f}{\partial t\partial s}(a,y)\right|^q + (1-t)(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q\right\} dtds\right)^{\frac{1}{q}} \\
=\; &\frac{1}{2^{\frac{2}{q}}}\left(\left|\frac{\partial^2 f}{\partial t\partial s}(x,y)\right|^q + \left|\frac{\partial^2 f}{\partial t\partial s}(x,c)\right|^q + \left|\frac{\partial^2 f}{\partial t\partial s}(a,y)\right|^q + \left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q\right)^{\frac{1}{q}}
\end{aligned} \tag{2.1}$$

and similarly, we get

$$\begin{aligned}
&\left(\int_0^1\int_0^1 \left|\frac{\partial^2 f}{\partial t\partial s}(tx+(1-t)a, sy+(1-s)d)\right|^q dsdt\right)^{\frac{1}{q}} \\
\leq\; &\frac{1}{2^{\frac{2}{q}}}\left(\left|\frac{\partial^2 f}{\partial t\partial s}(x,y)\right|^q + \left|\frac{\partial^2 f}{\partial t\partial s}(x,d)\right|^q + \left|\frac{\partial^2 f}{\partial t\partial s}(a,y)\right|^q + \left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q\right)^{\frac{1}{q}},
\end{aligned} \tag{2.2}$$

$$\begin{aligned}
&\left(\int_0^1\int_0^1 \left|\frac{\partial^2 f}{\partial t\partial s}(tx+(1-t)b, sy+(1-s)c)\right|^q dsdt\right)^{\frac{1}{q}} \\
\leq\; &\frac{1}{2^{\frac{2}{q}}}\left(\left|\frac{\partial^2 f}{\partial t\partial s}(x,y)\right|^q + \left|\frac{\partial^2 f}{\partial t\partial s}(x,c)\right|^q + \left|\frac{\partial^2 f}{\partial t\partial s}(b,y)\right|^q + \left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q\right)^{\frac{1}{q}},
\end{aligned} \tag{2.3}$$

$$\begin{aligned}
&\left(\int_0^1\int_0^1 \left|\frac{\partial^2 f}{\partial t\partial s}(tx+(1-t)b, sy+(1-s)d)\right|^q dsdt.\right)^{\frac{1}{q}} \\
\leq\; &\frac{1}{2^{\frac{2}{q}}}\left(\left|\frac{\partial^2 f}{\partial t\partial s}(x,y)\right|^q + \left|\frac{\partial^2 f}{\partial t\partial s}(x,d)\right|^q + \left|\frac{\partial^2 f}{\partial t\partial s}(b,y)\right|^q + \left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q\right)^{\frac{1}{q}}.
\end{aligned} \tag{2.4}$$

Then by using the inequalities (2.1)-(2.4) in (**??**), we obtain

$$\begin{aligned}
&\left|A+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(u,v)\,dudv\right| \\
\leq\ & \frac{1}{(p+1)^{\frac{2}{p}}}\frac{1}{2^{\frac{2}{q}}}\times \\
&\left\{\frac{(x-a)^2(y-c)^2}{(b-a)(d-c)}\left(\left|\frac{\partial^2 f}{\partial t\partial s}(x,y)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(x,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(a,y)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q\right)^{\frac{1}{q}}\right. \\
&+\frac{(x-a)^2(d-y)^2}{(b-a)(d-c)}\left(\left|\frac{\partial^2 f}{\partial t\partial s}(x,y)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(x,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(a,y)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q\right)^{\frac{1}{q}} \\
&+\frac{(b-x)^2(y-c)^2}{(b-a)(d-c)}\left(\left|\frac{\partial^2 f}{\partial t\partial s}(x,y)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(x,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,y)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q\right)^{\frac{1}{q}} \\
&\left.+\frac{(b-x)^2(d-y)^2}{(b-a)(d-c)}\left(\left|\frac{\partial^2 f}{\partial t\partial s}(x,y)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(x,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,y)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q\right)^{\frac{1}{q}}\right\}
\end{aligned}$$

which completes the proof. □

Corollary 2. *(1) Under the assumptions of Theorem 3, if we choose $x=a$, $y=c$, or $x=b$, $y=d$, we obtain the following inequality;*

$$\begin{aligned}
&\frac{1}{(b-a)(d-c)}\left|f(b,d)-(b-a)\int_c^d f(b,v)\,dv-(d-c)\int_a^b f(u,d)\,du+\int_a^b\int_c^d f(u,v)\,dudv\right| \\
\leq\ & \frac{1}{(b-a)(d-c)(p+1)^{\frac{2}{p}}2^{\frac{2}{q}}}\left(\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q\right)^{\frac{1}{q}}
\end{aligned}$$

(2)Under the assumptions of Theorem 3, if we choose $x=b$, $y=d$, we obtain the following inequality;

$$\begin{aligned}
&\frac{1}{(b-a)(d-c)}\left|f(a,c)-(b-a)\int_c^d f(a,v)\,dv-(d-c)\int_a^b f(u,c)\,du+\int_a^b\int_c^d f(u,v)\,dudv\right| \\
\leq\ & \frac{1}{(b-a)(d-c)(p+1)^{\frac{2}{p}}2^{\frac{2}{q}}}\left(\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q\right)^{\frac{1}{q}}
\end{aligned}$$

(3)Under the assumptions of Theorem 3, if we choose $x=a$, $y=d$, we obtain the following inequality;

$$\begin{aligned}
&\frac{1}{(b-a)(d-c)}\left|f(b,c)-(b-a)\int_c^d f(b,v)\,dv-(d-c)\int_a^b f(u,c)\,du+\int_a^b\int_c^d f(u,v)\,dudv\right| \\
\leq\ & \frac{1}{(b-a)(d-c)(p+1)^{\frac{2}{p}}2^{\frac{2}{q}}}\left(\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q\right)^{\frac{1}{q}}
\end{aligned}$$

(4) Under the assumptions of Theorem 3, if we choose $x = b$, $y = c$, we obtain the following inequality;

$$
\begin{aligned}
& \frac{1}{(b-a)(d-c)}\left| f(a,d) - (b-a)\int_c^d f(a,v)\,dv - (d-c)\int_a^b f(u,d)\,du + \int_a^b\int_c^d f(u,v)\,dudv \right| \\
\leq\; & \frac{1}{(b-a)(d-c)(p+1)^{\frac{2}{p}}2^{\frac{2}{q}}}\left(\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q + \left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q + \left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q + \left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q \right)^{\frac{1}{q}}
\end{aligned}
$$

(5) Under the assumptions of Theorem 3, if we choose $x = \frac{a+b}{2}$, $y = \frac{c+d}{2}$, we obtain the following inequality;

$$
\begin{aligned}
& \left| \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4(b-a)(d-c)} - \frac{1}{2(d-c)}\int_c^d f(a,v)\,dv - \frac{1}{2(d-c)}\int_c^d f(b,v)\,dv \right. \\
& \left. - \frac{1}{2(b-a)}\int_a^b f(u,d)\,du - \frac{1}{2(b-a)}\int_a^b f(u,c)\,du + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(u,v)\,dudv \right| \\
\leq\; & \frac{1}{(b-a)(d-c)\,16\,(p+1)^{\frac{2}{p}}2^{\frac{2}{q}}} \times \\
& \left\{ \left(\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q + \left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},c\right)\right|^q + \left|\frac{\partial^2 f}{\partial t\partial s}\left(a,\frac{c+d}{2}\right)\right|^q + \left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q \right)^{\frac{1}{q}} \right. \\
& + \left(\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q + \left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},d\right)\right|^q + \left|\frac{\partial^2 f}{\partial t\partial s}\left(a,\frac{c+d}{2}\right)\right|^q + \left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q \right)^{\frac{1}{q}} \\
& + \left(\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q + \left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},c\right)\right|^q + \left|\frac{\partial^2 f}{\partial t\partial s}\left(b,\frac{c+d}{2}\right)\right|^q + \left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q \right)^{\frac{1}{q}} \\
& \left. + \left(\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q + \left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},d\right)\right|^q + \left|\frac{\partial^2 f}{\partial t\partial s}\left(b,\frac{c+d}{2}\right)\right|^q + \left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q \right)^{\frac{1}{q}} \right\}.
\end{aligned}
$$

Theorem 4. *Let $f : \Delta = [a,b]\times[c,d] \to \mathbb{R}$ be a partial differentiable mapping on $\Delta = [a,b]\times[c,d]$ and $\frac{\partial^2 f}{\partial t\partial s} \in L(\Delta)$. If $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$, $q \geq 1$, is a convex function on the*

co-ordinates on Δ, *then the following inequality holds;*

$$\begin{aligned}
&\left|A+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(u,v)\,dudv\right| \\
\leq \left(\frac{1}{4}\right)^{1-\frac{1}{q}} &\left\{K\left(\frac{1}{36}\left|\frac{\partial^2 f}{\partial t\partial s}(x,y)\right|^q+\frac{1}{18}\left|\frac{\partial^2 f}{\partial t\partial s}(x,c)\right|^q+\frac{1}{18}\left|\frac{\partial^2 f}{\partial t\partial s}(a,y)\right|^q+\frac{1}{9}\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q\right)^{\frac{1}{q}}\right. \\
&+L\left(\frac{1}{36}\left|\frac{\partial^2 f}{\partial t\partial s}(x,y)\right|^q+\frac{1}{18}\left|\frac{\partial^2 f}{\partial t\partial s}(x,d)\right|^q+\frac{1}{18}\left|\frac{\partial^2 f}{\partial t\partial s}(a,y)\right|^q+\frac{1}{9}\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q\right)^{\frac{1}{q}} \\
&+M\left(\frac{1}{36}\left|\frac{\partial^2 f}{\partial t\partial s}(x,y)\right|^q+\frac{1}{18}\left|\frac{\partial^2 f}{\partial t\partial s}(x,c)\right|^q+\frac{1}{18}\left|\frac{\partial^2 f}{\partial t\partial s}(b,y)\right|^q+\frac{1}{9}\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q\right)^{\frac{1}{q}} \\
&\left.+N\left(\frac{1}{36}\left|\frac{\partial^2 f}{\partial t\partial s}(x,y)\right|^q+\frac{1}{18}\left|\frac{\partial^2 f}{\partial t\partial s}(x,d)\right|^q+\frac{1}{18}\left|\frac{\partial^2 f}{\partial t\partial s}(b,y)\right|^q+\frac{1}{9}\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q\right)^{\frac{1}{q}}\right\}
\end{aligned}$$

where

$$\begin{aligned}
K &= \frac{(x-a)^2(y-c)^2}{(b-a)(d-c)} \\
L &= \frac{(x-a)^2(d-y)^2}{(b-a)(d-c)} \\
M &= \frac{(b-x)^2(y-c)^2}{(b-a)(d-c)} \\
N &= \frac{(b-x)^2(d-y)^2}{(b-a)(d-c)}.
\end{aligned}$$

Proof. From Lemma 1, we have

$$\begin{aligned}
&\left|A+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(u,v)\,dudv\right| \\
\leq\ & \frac{(x-a)^2(y-c)^2}{(b-a)(d-c)}\int_0^1\int_0^1|(t-1)(s-1)|\left|\frac{\partial^2 f}{\partial t\partial s}(tx+(1-t)a,sy+(1-s)c)\right|dsdt \\
&+\frac{(x-a)^2(d-y)^2}{(b-a)(d-c)}\int_0^1\int_0^1|(t-1)(1-s)|\left|\frac{\partial^2 f}{\partial t\partial s}(tx+(1-t)a,sy+(1-s)d)\right|dsdt \\
&+\frac{(b-x)^2(y-c)^2}{(b-a)(d-c)}\int_0^1\int_0^1|(1-t)(s-1)|\left|\frac{\partial^2 f}{\partial t\partial s}(tx+(1-t)b,sy+(1-s)c)\right|dsdt \\
&+\frac{(b-x)^2(d-y)^2}{(b-a)(d-c)}\int_0^1\int_0^1|(1-t)(1-s)|\left|\frac{\partial^2 f}{\partial t\partial s}(tx+(1-t)b,sy+(1-s)d)\right|dsdt.
\end{aligned}$$

By using the well known Power mean inequality for double integrals, then one has:

$$
\begin{aligned}
(2.5)\quad & \left| A + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(u,v)\, dudv \right| \\
\leq\ & \frac{(x-a)^2 (y-c)^2}{(b-a)(d-c)} \left(\int_0^1 \int_0^1 |(t-1)(s-1)|\, dsdt \right)^{1-\frac{1}{q}} \\
& \times \left(\int_0^1 \int_0^1 |(t-1)(s-1)| \left| \frac{\partial^2 f}{\partial t \partial s}(tx+(1-t)a, sy+(1-s)c) \right|^q dsdt \right)^{\frac{1}{q}} \\
& + \frac{(x-a)^2 (d-y)^2}{(b-a)(d-c)} \left(\int_0^1 \int_0^1 |(t-1)(1-s)|\, dsdt \right)^{1-\frac{1}{q}} \\
& \times \left(\int_0^1 \int_0^1 |(t-1)(1-s)| \left| \frac{\partial^2 f}{\partial t \partial s}(tx+(1-t)a, sy+(1-s)d) \right|^q dsdt \right)^{\frac{1}{q}} \\
& + \frac{(b-x)^2 (y-c)^2}{(b-a)(d-c)} \left(\int_0^1 \int_0^1 |(1-t)(s-1)|\, dsdt \right)^{1-\frac{1}{q}} \\
& \times \left(\int_0^1 \int_0^1 |(1-t)(s-1)| \left| \frac{\partial^2 f}{\partial t \partial s}(tx+(1-t)b, sy+(1-s)c) \right|^q dsdt \right)^{\frac{1}{q}} \\
& + \frac{(b-x)^2 (d-y)^2}{(b-a)(d-c)} \left(\int_0^1 \int_0^1 |(1-t)(1-s)|\, dsdt \right)^{1-\frac{1}{q}} \\
& \times \left(\int_0^1 \int_0^1 |(1-t)(1-s)| \left| \frac{\partial^2 f}{\partial t \partial s}(tx+(1-t)b, sy+(1-s)d) \right|^q dsdt. \right)^{\frac{1}{q}}
\end{aligned}
$$

Since $\left|\frac{\partial^2 f}{\partial t \partial s}\right|^q$ is convex function on the co-ordinates on Δ, we know that for $t, s \in [0,1]$

$$
\begin{aligned}
& \left| \frac{\partial^2 f}{\partial t \partial s}(tx+(1-t)a, sy+(1-s)c) \right|^q \\
\leq\ & t \left| \frac{\partial^2 f}{\partial t \partial s}(x, sy+(1-s)c) \right|^q + (1-t) \left| \frac{\partial^2 f}{\partial t \partial s}(a, sy+(1-s)c) \right|^q \\
\leq\ & t \left(s \left| \frac{\partial^2 f}{\partial t \partial s}(x,y) \right|^q + (1-s) \left| \frac{\partial^2 f}{\partial t \partial s}(x,c) \right|^q \right) + (1-t) \left(s \left| \frac{\partial^2 f}{\partial t \partial s}(a,y) \right|^q + (1-s) \left| \frac{\partial^2 f}{\partial t \partial s}(a,c) \right|^q \right)
\end{aligned}
$$

and by using the fact that

$$\left(\int_0^1\int_0^1 |(t-1)(s-1)|\,dsdt\right)^{1-\frac{1}{q}} = \left(\frac{1}{4}\right)^{1-\frac{1}{q}}$$

we get

$$\begin{aligned}
(2.6) & \left(\int_0^1\int_0^1 |(t-1)(s-1)| \left|\frac{\partial^2 f}{\partial t\partial s}(tx+(1-t)a, sy+(1-s)c)\right|^q dsdt\right)^{\frac{1}{q}} \\
\leq & \left(\int_0^1\int_0^1 |(t-1)(s-1)| \left\{ts\left|\frac{\partial^2 f}{\partial t\partial s}(x,y)\right|^q + t(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(x,c)\right|^q\right.\right. \\
& \left.\left. +(1-t)s\left|\frac{\partial^2 f}{\partial t\partial s}(a,y)\right|^q + (1-t)(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q\right\} dtds\right)^{\frac{1}{q}} \\
= & \left(\frac{1}{36}\left|\frac{\partial^2 f}{\partial t\partial s}(x,y)\right|^q + \frac{1}{18}\left|\frac{\partial^2 f}{\partial t\partial s}(x,c)\right|^q + \frac{1}{18}\left|\frac{\partial^2 f}{\partial t\partial s}(a,y)\right|^q + \frac{1}{9}\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q\right)^{\frac{1}{q}}
\end{aligned}$$

and similarly, we get

$$\begin{aligned}
(2.7) & \left(\int_0^1\int_0^1 \left|\frac{\partial^2 f}{\partial t\partial s}(tx+(1-t)a, sy+(1-s)d)\right|^q dsdt\right)^{\frac{1}{q}} \\
\leq & \left(\frac{1}{36}\left|\frac{\partial^2 f}{\partial t\partial s}(x,y)\right|^q + \frac{1}{18}\left|\frac{\partial^2 f}{\partial t\partial s}(x,d)\right|^q + \frac{1}{18}\left|\frac{\partial^2 f}{\partial t\partial s}(a,y)\right|^q + \frac{1}{9}\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q\right)^{\frac{1}{q}},
\end{aligned}$$

$$\begin{aligned}
(2.8) & \left(\int_0^1\int_0^1 \left|\frac{\partial^2 f}{\partial t\partial s}(tx+(1-t)b, sy+(1-s)c)\right|^q dsdt\right)^{\frac{1}{q}} \\
\leq & \left(\frac{1}{36}\left|\frac{\partial^2 f}{\partial t\partial s}(x,y)\right|^q + \frac{1}{18}\left|\frac{\partial^2 f}{\partial t\partial s}(x,c)\right|^q + \frac{1}{18}\left|\frac{\partial^2 f}{\partial t\partial s}(b,y)\right|^q + \frac{1}{9}\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q\right)^{\frac{1}{q}},
\end{aligned}$$

$$\begin{aligned}
(2.9) & \left(\int_0^1\int_0^1 \left|\frac{\partial^2 f}{\partial t\partial s}(tx+(1-t)b, sy+(1-s)d)\right|^q dsdt.\right)^{\frac{1}{q}} \\
\leq & \left(\frac{1}{36}\left|\frac{\partial^2 f}{\partial t\partial s}(x,y)\right|^q + \frac{1}{18}\left|\frac{\partial^2 f}{\partial t\partial s}(x,d)\right|^q + \frac{1}{18}\left|\frac{\partial^2 f}{\partial t\partial s}(b,y)\right|^q + \frac{1}{9}\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q\right)^{\frac{1}{q}}.
\end{aligned}$$

Then by using the inequalities (2.6)-(2.9) in (2.5), we obtain

$$\begin{aligned}
&\left| A+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(u,v)\,dudv\right| \\
\leq \left(\frac{1}{4}\right)^{1-\frac{1}{q}} &\left\{ K\left(\frac{1}{36}\left|\frac{\partial^2 f}{\partial t\partial s}(x,y)\right|^q+\frac{1}{18}\left|\frac{\partial^2 f}{\partial t\partial s}(x,c)\right|^q+\frac{1}{18}\left|\frac{\partial^2 f}{\partial t\partial s}(a,y)\right|^q+\frac{1}{9}\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q\right)^{\frac{1}{q}}\right. \\
&+L\left(\frac{1}{36}\left|\frac{\partial^2 f}{\partial t\partial s}(x,y)\right|^q+\frac{1}{18}\left|\frac{\partial^2 f}{\partial t\partial s}(x,d)\right|^q+\frac{1}{18}\left|\frac{\partial^2 f}{\partial t\partial s}(a,y)\right|^q+\frac{1}{9}\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q\right)^{\frac{1}{q}} \\
&+M\left(\frac{1}{36}\left|\frac{\partial^2 f}{\partial t\partial s}(x,y)\right|^q+\frac{1}{18}\left|\frac{\partial^2 f}{\partial t\partial s}(x,c)\right|^q+\frac{1}{18}\left|\frac{\partial^2 f}{\partial t\partial s}(b,y)\right|^q+\frac{1}{9}\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q\right)^{\frac{1}{q}} \\
&\left.+N\left(\frac{1}{36}\left|\frac{\partial^2 f}{\partial t\partial s}(x,y)\right|^q+\frac{1}{18}\left|\frac{\partial^2 f}{\partial t\partial s}(x,d)\right|^q+\frac{1}{18}\left|\frac{\partial^2 f}{\partial t\partial s}(b,y)\right|^q+\frac{1}{9}\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q\right)^{\frac{1}{q}}\right\}
\end{aligned}$$

which completes the proof. □

References

[1] S.S. Dragomir, On Hadamard's inequality for convex functions on the co-ordinates in a rectangle from the plane, Taiwanese Journal of Math., 5, 2001, 775-788.

[2] M. Alomari and M. Darus, On the Hadamard's inequality for log −convex functions on the co-ordinates, Journal of Inequalities and Appl., 2009, article ID 283147.

[3] M.K. Bakula and J. Pečarić, On the Jensen's inequality for convex functions on the co-ordinates in a rectangle from the plane, Taiwanese Journal of Math., 5, 2006, 1271-1292.

[4] M.E. Özdemir, E. Set, M.Z. Sarıkaya, Some new Hadamard's type inequalities for co-ordinated $m-$convex and $(\alpha, m)-$convex functions, Hacettepe J. of. Math. and St., 40, 219-229, (2011).

[5] M.Z. Sarıkaya, E. Set, M. Emin Özdemir and S.S. Dragomir, New some Hadamard's type inequalities for co-ordinated convex functions, Accepted.

[6] M. Emin Özdemir, Havva Kavurmacı, Ahmet Ocak Akdemir and Merve Avcı, Inequalities for convex and $s-$convex functions on $\Delta = [a, b]x[c, d]$, Journal of Inequalities and Applications 2012:20, doi:10.1186/1029-242X-2012-20.

[7] M. Emin Özdemir, M. Amer Latif and Ahmet Ocak Akdemir, On some Hadamard-type inequalities for product of two $s-$convex functions on the co-ordinates, Journal of Inequalities and Applications, 2012:21, doi:10.1186/1029-242X-2012-21.

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